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3D consistent boundary-flux problem in domains with complex geometry

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Abstract

Purpose – To obtain error estimates for 3D consistent boundary-flux approximations.

Design/methodology/approach – Isoparametric approach is used for constructing finite-element approximations.

Findings – This research study presents a convergence analysis of 3D boundary-flux approximations. Error estimates are proved for the approximate solutions of the problem under consideration.

Research limitations/implications – General results for a consistent boundary-flux problem are obtained for all 3D domains with Lipschitz-continuous boundary. This investigation will be continued studying combined effect of curved boundaries and isoparametric numerical integration. An optimal refined strategy with respect to algorithmic aspects for solving 3D boundary-flux problem also will be considered.

Practical implications – The obtained results enable engineers to calculate the flux across the curved boundaries using finite element method (FEM).

Originality/value – The paper presents an isoparametric finite-element method for a 3D consistent boundary-flux problem in domains with complex geometry. The work is addressed to the possible-related fields of interest of postgraduate students and specialists in fluid mechanics and numerical analysis.

Keywords Finite element analysis, Boundary-elements methods, Approximation theory

Paper type Research paper

1. Introduction

Computation of boundary-flux is well motivated concerning various engineering problems, for instance the classical drift-diffusion model, determination of the behavior of the body immersed in a viscous incompressible fluid, obtaining of stress intensity factor, moments of a shell or plate, heat and mass transfer, potential flow, magnetostatics, elasticity problems, etc.

The finite element method (FEM) is among the most powerful tools for solving boundary value problems and, in particular, for solving boundary-flux problem. Early applications of FEM for finding boundary-flux approximations are based on the idea proposed by Wheeler (1973) and developed by Carey (1982) (Carey *et al.*, 1985). Confine to polygonal domains and affine FEM Barret and Elliot (1987) proved an asymptotic error $O(h^{n-1/2})$ in approximate flux, where n is the degree of trial functions.

Various postprocessing techniques for increasing the rate of convergence of the boundary-flux approximations are developed by Douglas *et al.* (1974), Lazarov and Pehlivanov (1989) and Pehlivanov *et al.* (1992). More recent results concerning boundary-flux computations are obtained by Carey (2002), Chipot and Rougirel (2001),

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Huang and Zhong (2004), Zheng and Song (2004), etc. The above results are obtained in polygonal domains. Problems in curved domains need completely different approach, namely isoparametric approach. Optimal convergence rate for the boundary-flux approximations in two-dimensional domains with complex geometry was proved by Andreev and Todorov (2005). Here the optimality is in the sense that isoparametric approximations have the same rate of convergence as the ones in the affine case where polygonal domains are considered.

Lenoir (1986), Brenner and Scott (1994) and Andreev and Todorov (2005) give comparisons between the bilinear forms only in the case when $L = \Delta$. Comparisons between bilinear forms arising from general second-order elliptic operator are presented here.

The present investigation deals with a FEM for 3D boundary flux problem in a curved domain with Lipschitz-continuous boundary. The paper is organized as follows. The weak formulation of the boundary-flux problem is compiled in Section 2. Finite element discretizations and some basic properties of Lenoir map are described in Section 3. Optimal convergence order for the boundary-flux approximations is proved in Section 4. This is the main result in the present investigation.

2. Setting of the problem

Let $\Omega \subset \mathbf{R}^3$ be a bounded curved domain with Lipschitz-continuous boundary Γ. Define a Dirichlet problem:

$$\mathcal{P}: \begin{cases} \text{find a functioin } u \text{ satisfying} \\ Lu = f \text{ in } \Omega, \\ u = 0 \text{ on } \Gamma \end{cases}$$

The map:

$$Lu = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(a_{ij}\frac{\partial}{\partial x_i}\right)$$

is a linear operator with $a_{ij} \in C^1(\Omega),\ i,j = 1,2,3$. Assume that the matrix $A = (a_{ij}(x))_{i,j\in\{1,2,3\}}$ is uniformly positive definite in Ω, i.e. the operator L is strongly elliptic.

Standard notations for the Sobolev spaces (Ciarlet, 1978) and associated norms and seminorms are used throughout this consideration. Define the Sobolev space:

$$H_0^1(\Omega) = \{v \in H^1(\Omega) \,|\, v = 0 \text{ on } \Gamma\},$$

the bilinear form:

$$a(u,v) = \int_\Omega \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\mathrm{d}x, \quad u,v \in \mathbf{V} = H_0^1(\Omega)$$

and the linear functional:

$$(f,v) = \int_{\Omega} fv\,\mathrm{d}x, \quad v \in \mathbf{V}.$$

The bilinear form $a(\cdot,\cdot)$ is coercive on $\mathbf{V}\times\mathbf{V}$, since L is strongly elliptic. The boundedness of a_{ij} on $\bar{\Omega}$ implies that $a(\cdot,\cdot)$ is continuous on $H^1(\Omega)$.

Write the weak problem $\mathcal{P}_W$ associated with $\mathcal{P}$:

$$\mathcal{P}_W : \begin{cases} \text{find a function } u \in \mathbf{V} \text{ such that} \\ a(u,v) = (f,v) \ \ \forall v \in \mathbf{V}. \end{cases}$$

Introduce the usual hypotheses concerning the smoothness of the weak solution:

C1. The boundary Γ is piecewise C^{n+1}, $n \geq 2$.

C2. The right hand side $f \in W^{n,\infty}(\Omega)$ and the weak solution $u \in H^{n+1}(\Omega)$.

C3. The coefficients $a_{ij} \in W^{n,\infty}(\Omega)$.

Define the normal flux q across boundary Γ by:

$$q = \underline{\sigma}\cdot\underline{n} = -\sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial u}{\partial x_i}\cos(\underline{n}, x_j), \quad x \in \Gamma,$$

where u is the solution of $\mathcal{P}_W$, $\underline{\sigma} = -A^t(\nabla u)$ is a vector function, n is the outward normal vector to the boundary Γ and "t" is the sign for transposition.

Define the weak boundary-flux problem as follows:

$$\mathcal{F}_W : \begin{cases} \text{find a function } q \in H^{1/2}(\Gamma) \text{ such that} \\ - < q,v >= a(u,v) - (f,v) \ \ \forall\, v \in H^1(\Omega), \end{cases}$$

where $< \cdot,\cdot >$ denotes the inner product on the boundary, i.e.

$$< q,v >= \int_{\Gamma} qv\,\mathrm{d}s.$$

3. Finite element discretizations

This section dwells largely on tetrahedral triangulations of curved domains. Usually, when solving a problem in a curved domain Ω we separate some "nice" domain $\hat{\Omega} \subset \Omega$, which can be triangulated by straight elements. The domain $\hat{\Omega}$ is as big as possible. Triangulate the rest of Ω by curved elements.

Rewrite some basic definitions concerning finite element triangulations. Assume that any finite element $K \in \tau_h$ is generated by invertible isoparametric finite element transformation F_K defined on one and the same element $(\hat{T}, \hat{P}, \hat{\Sigma})$ called reference finite element.

Define the reference finite element as follows:

$$\hat{T} = \left\{ \hat{x} = (\hat{x}_1, \hat{x}_2, \hat{x}_2) | \hat{x}_i \geq 0, \ \ i = 1,2,3, \ \ \sum_{i=1}^{3} \hat{x}_i \leq 1 \right\}$$

is the canonical 3-simplex; $\hat{P} = P_n(\hat{T})$, where P_n is the space of all polynomials of degree, not exceeding n:

$$\hat{\Sigma} = \left\{ \hat{a} | \hat{a}_i = \frac{k_i}{n},\ i = 1,2,3,\ \sum_{i=1}^{3} k_i \leq n;\ k_i \in \mathbf{N} \cup \{0\},\ i = 1,2,3 \right\}$$

is the set of all Lagrangian interpolation nodes of order n.

The boundary layer of any triangulation τ_h consists of those elements, which have more than one vertex on the boundary. Unifying all the elements in triangulation τ_h we obtain an approximate domain $\Omega_h = \cup_{K \in \tau_h} K$ with boundary Γ_h.

Define the finite element space $\mathbf{V}_h$ by:

$$\mathbf{V}_h = \{ v_h \in C(\Omega_h) | v_{h|K} \in P_K,\ K \in \tau_h \},$$

where $P_K = \{ p : K \to \mathbf{R} | p = \hat{p} \circ F_K^{-1},\ \hat{p} \in \hat{P} \}$. It is well known that $\mathbf{V}_h \subset H^1(\Omega_h)$ if the triangulation τ_h is regular.

If a map $F(x)$ is k-times differentiable, we denote the k-th Fréchet derivative of $F(x)$ by $D^k \mathrm{F}(x)$. Let $\mathcal{L}_n\,(\mathbf{R}^3; \mathbf{R}^3)$ be the space of continuous n-linear mappings from $(\mathbf{R}^3)^n$ to $\mathbf{R}^3$ and $\hat{K}, \check{K}$ be bounded subsets of $\mathbf{R}^3$. For estimating the Fréchet derivatives and Jacobians we need the following seminorms:

$$|F|_{n,\infty,\hat{K}} = \sup_{\hat{x} \in \hat{K}} \| D^n F(\hat{x}) \|_{\mathcal{L}_n(\mathrm{R}^3;\mathrm{R}^3)},$$
$$|F^{-1}|_{n,\infty,\check{K}} = \sup_{\check{x} \in \check{K}} \| D^n F^{-1}(\check{x}) \|_{\mathcal{L}_n(\mathrm{R}^3;\mathrm{R}^3)}, \quad n = 0,1,2,\ldots$$

for arbitrary sufficiently smooth transformation $F : \hat{K} \to \check{K}$ with sufficiently smooth inverse transformation F^{-1}.

Further, we shall apply the construction of n-regular isoparametric triangulation τ_h presented by Lenoir (1986).

Let Φ_h: $\Omega_h \to \Omega$ be the invertible mapping obtained by Lenoir (1986) and ϕ_h:$\Gamma_h \to \Gamma$ be its restriction on the boundary Γ_h. The map Φ_h plays an important role in our analysis, therefore we present some basic features of this mapping. The map Φ_h:

- is equal to the identity map on elements, which do not belong to the boundary layer;
- have the property that the distance from any point on Γ to the closest point on Γ_h is Ch^{n+1} at most; and
- has the following estimates for the Fréchet derivatives and Jacobians:

$$|J(\Phi_h)|_{0,\infty,\Omega_h} = O(1), \quad |J(\Phi_h^{-1})|_{0,\infty,\Omega} = O(1), \tag{1}$$

$$|\Phi_h|_{1,\infty,\Omega_h} = O(1), \quad |\Phi_h^{-1}|_{1,\infty,\Omega} = O(1), \tag{2}$$

and:

$$|D(I - \Phi_h)|_{0,\infty,\Omega_h} = O(h^n), \quad |D(I - (\Phi_h^{-1})|_{0,\infty,\Omega} = O(h^n), \tag{3}$$

$$|J(\Phi_h) - 1|_{0,\infty,\Omega_h} = O(h^n), \quad |J(\Phi_h^{-1}) - 1|_{0,\infty,\Omega} = O(h^n) \tag{4}$$

where I is the identity map and $J(\Phi_h)$ is the Jacobian of Φ_h.

We shall use the following finite dimensional spaces associated with the triangulation τ_h:

$$v_h = \{v_h \in \mathbf{V}_h | v_h = 0 \text{ at the edges of } \Omega\},$$

$$\mathbf{B}_h = \{\omega_h | \omega_h = v_h | \Gamma_h, v_h \in \mathscr{V}_h\},$$

$$\mathbf{V}_0^h = \{v_h \in \mathbf{V}_h | v_h = 0 \text{ on the boundary } \Gamma_h\},$$

$$\check{\mathbf{V}}_h = \{\check{v}_h \in v_h \circ \Phi_h^{-1} | v_h \in \mathbf{V}_h\}.$$

Analogously $\check{\mathbf{V}}_0^h$ consists of those functions from $\check{\mathbf{V}}_h$, which are zero on the boundary Γ. A space similar to $\mathscr{V}_h$ is used by Pehlivanov *et al.* (1992) in 2D case where the functions v_h are zero at the corners of Ω. We shall use also the space:

$$\mathbf{V}^* = \{v^* = v \circ \Phi_h | v \in H^1(\Omega)\}.$$

Let I_h: $H^1(\Omega) \rightarrow V_h$ be a standard interpolation operator on the whole triangulation τ_h. Write the approximating bilinear form and L^2-scalar product in $\mathbf{V}_h$:

$$a_h(u_h, v_h) = \int_{\Omega_h} \sum_{i,j=1}^{3} a_{ij}^h(x) \frac{\partial u_h}{\partial x_i} \frac{\delta v_h}{\partial x_j} \mathrm{d}x \quad \forall u_h, v_h \in \mathbf{V}_h, \tag{5}$$

$$(u_h, v_h)_h = \int_{\Omega_h} u_h v_h \, \mathrm{d}x \quad \forall u_h, v_h \in \mathbf{V}_h,$$

where $a_{ij}^h = I_h(a_{ij}^*)$.

Definition 1. The bilinear forms (equation (5)) are called uniformly $\mathbf{V}_0^h$-elliptic, if there exists a constant $\varepsilon > 0$ independent of the spaces $\mathbf{V}_0^h$, such that for all h sufficiently small:

$$\varepsilon \|v_h\|_{1,\Omega_h}^2 \leq a_h(v_h, v_h) \quad \forall v_h \in \mathbf{V}_0^h.$$

Compile the discrete problem $\mathcal{P}_h$ corresponding to the problem $\mathcal{P}_W$:

$$\mathcal{P}_h : \begin{cases} \text{find } u_h \in \mathbf{V}_0^h \text{ such that} \\ a_h(u_h, v_h) = (f_h, v_h)_h \quad \forall u_h \in \mathbf{V}_0^h, \end{cases}$$

where $f_h = I_h(f^*)$.

Assume that the solution u_h of the problem $\mathcal{P}_h$ is already found. Then we can construct the approximate boundary-flux problem:

$$\mathscr{F}_h : \begin{cases} \text{find } q_h \in B_h \text{ such that} \\ - < q_h, v_h >_h = a_h(u_h, v_h) - (f_h, v_h)_h \quad \forall v_h \in V_h \end{cases},$$

where:

$$< q_h, v_h >_h = \int_{\Gamma_h} q_h v_h \, dS.$$

For the first time a problem similar to $\mathcal{F}_h$ is considered by Carey *et al.* (1985) in the 2D consistent case.

Notations $C, C_1, C_2, \ldots$ are reserved for generic positive constants, which may vary with the context.

4. Error estimates

Convergence analysis is the point of interest in this section. Estimates of the error in the boundary-flux approximations are obtained.

The uniformly $\mathbf{V}_0^h$-ellipticity is a very important property of the bilinear forms (equation (5)), because this property ensures a unique solution of the problem $\mathcal{P}_h$. Since, $a_h(\cdot,\cdot)$ contains variable coefficients we should prove the following lemma.

Lemma 1. Let the triangulation τ_h be n-regular in the sense of Ciarlet and Raviart (1972) and $\varepsilon_L > 0$ be the ellipticity constant of the operator L. Then there exists a constant h_0 for any ε:$0 < \varepsilon < \varepsilon_L$ such that:

$$\varepsilon \|v_h\|^2_{1,\Omega_h} \leq a_h(v_h, v_h) \quad \forall h \leq h_0 \text{ and } \forall v_h \in \mathbf{V}_0^h.$$

Proof. Define the n-th norm of the matrix A by:

$$\|A\|_n = \max_{i,j \in \{1,2,3\}} \|a_{ij}\|_{n,\infty,\Omega}.$$

Applying Cauchy inequality and $y \stackrel{\text{def}}{=} \Phi_h(x), x \in \Omega_h$ we have:

$$\begin{aligned}
\sum_{i,j=1}^{3} a_{ij}^h(x)\xi_i\xi_j &= \sum_{i,j=1}^{3} \left(a_{ij}^h(x) - a_{ij}(y)\right)\xi_i\xi_j + \sum_{i,j=1}^{3} a_{ij}(y)\xi_i\xi_j \\
&= \sum_{i,j=1}^{3} a_{ij}(y)\xi_i\xi_j + \sum_{i,j=1}^{3} \left(I_h(a_{ij}^*) - a_{ij}^*\right)(x)\xi_i\xi_j \\
&\geq \varepsilon_L \sum_{i,j=1}^{3} \xi_i^2 - \sum_{i,j=1}^{3} \left|I_h(a_{ij}^*) - a_{ij}^*\right|_{0,\infty,\Omega_h} \xi_i\xi_j \\
&\geq \varepsilon_L \sum_{i,j=1}^{3} \xi_i^2 - Ch^n \sum_{i,j=1}^{3} \|a_{ij}^*\|_{n,\infty,\Omega_h} \xi_i\xi_j \\
&\geq (\varepsilon_L - Ch^n \|A\|_n) \sum_{i=1}^{3} \xi_i^2
\end{aligned}$$

for all $x \in \Omega_h$ and for all $\xi \in \mathbf{R}^3$.

Denote the Poincaré constant by $C(\Omega)$. Choosing:

$$h \leq h_0 = \sqrt[n]{\frac{\varepsilon_L - \varepsilon(1 + C(\Omega))}{C\|A\|_n} n}$$

we obtain:

$$\sum_{i,j=1}^{3} a_{ij}^{h}(x)\xi_i\xi_j \geq \varepsilon(1 + C(\Omega))\sum_{i,j=1}^{3} \xi_i^2.$$

We complete the proof using Poincaré inequality:

$$a_h(v_h, v_h) \geq \varepsilon(+C(\Omega))\int_{\Omega_h} \nabla v_h \cdot \nabla v_h \,\mathrm{d}x = \varepsilon(1 + C(\Omega))|v_h|^2_{1,\Omega_h} \geq \varepsilon\|v_h\|^2_{1,\Omega_h} \cdot$$

□

Lemma 1 ascertains that if L is strongly elliptic then the bilinear forms $a_h(\cdot,\cdot)$ are uniformly $\mathbf{V}_0^h$-elliptic.
A comparison between the bilinear forms $a_h(\cdot,\cdot)$ and $a(\cdot,\cdot)$ contains in the following lemma.

Lemma 2. The estimate:

$$|a(w, \check{v}_h) - a_h(w^*, v_h)| \leq Ch^n\|A\|_n\|w\|_{1,\Omega}|v_h|_{1,\Omega_h} \tag{6}$$

is valid $\forall w \in \mathbf{V}$ and $\forall v_h \in \varsigma_h$ if the triangulation τ_h be n-regular.

Proof. Adding and subtracting some terms in the left hand side of equation (6) we obtain:

$$\begin{aligned}
&|a(w, \check{v}_h) - a_h(w^*, v_h)| \\
&= \left| \int_{\Omega_h} \sum_{i,j=1}^{3} a_{ij}^{*}(x)(\nabla w^*(D\Phi_h)^{-1}\cdot \underline{e}_j)(\nabla v_\mathrm{h}(D\Phi_h)^{-1}\cdot \underline{e}_j)J(\Phi_h)\mathrm{d}x \right. \\
&\quad \left. - \int_{\Omega_h} \sum_{i,j=1}^{3} a_{ij}^{h}(x)\frac{\partial w^*}{\partial x_i}\frac{\partial v_h}{\partial x_j}\,\mathrm{d}x \right| \leq Ch^n\|A\|_n\|w\|_{1,\Omega}\|v_h\|_{1,\Omega_h} \\
&\quad + \left| \int_{\Omega_h} \sum_{i,j=1}^{3} a_{ij}^{h}(x)\left[(\nabla w^*(D\Phi_h)^{-1}\cdot \underline{e}_i)(\nabla v_h(D\Phi_h)^{-1}\cdot \underline{e}_j)J(\Phi_h) - \frac{\partial w^*}{\partial x_i}\frac{\partial v^h}{\partial x_j}\right]\mathrm{d}x \right|,
\end{aligned}$$

where $\{\underline{e}_i,\ i = 1, 2, 3\}$ is the canonical basis in $\mathbf{R}^3$. We use the notation $A \simeq B$ to indicate that the quantities A and B are uniformly equivalent with respect to the mesh parameter h. Extending the validity of Proposition 4 by Lenoir (1986) to the whole triangulation τ_h we obtain:

$$\|f\|_{m,\Omega} \simeq \|f^*\|_{m,\Omega_h}, \quad 0 \leq m \leq n \qquad \forall f \in H^{n+1}(\Omega), \tag{7}$$

$$\|f\|_{m,\Gamma} \simeq \|f\circ\phi_h\|_{m,\Gamma_h}, \quad 0 \leq m \leq n - 1/2 \qquad \forall f \in H^{n-1/2}(\Gamma). \tag{8}$$

Then:

$$
\begin{aligned}
|a(w,\check{v}_h) - a_h(w^*,v_h)| \le{}& Ch^n\|A\|_n\|w\|_{1,\Omega}|v_h|_{1,\Omega_h} \\
&+\left|\int_{\Omega_h}\sum_{i,j=1}^{3} a_{ij}^h(x)\Big(\nabla w^*(D\Phi_h)^{-1}\cdot \underline{e}_i\Big)\right. \\
&\left.\times\Big(\nabla v_h(D\Phi_h)^{-1}\cdot \underline{e}_j\Big)(J(\Phi_h-1)\mathrm{d}x\right| \\
&+\left|\int_{\Omega_h}\sum_{i,j=1}^{3} a_{ij}^h(x)\Big[\Big(\nabla w^*(D\Phi_h)^{-1}\cdot \underline{e}_i\Big)\right. \\
&\left.\times\Big(\nabla v_h(D\Phi_h)^{-1}\cdot \underline{e}_j\Big) - (\nabla w^*\cdot \underline{e}_i)(\nabla v_h\cdot \underline{e}_j)\Big]\mathrm{d}x\right| \\
\le{}& C\|A\|_n\|w\|_{1,\Omega}|v_h|_{1,\Omega_h}(h^n + \|J(\Phi_h)-1\|_{0,\infty,\Omega_h}) \\
&+\left|\int_{\Omega_h}\sum_{i,j=1}^{3} a_{ij}^h(x)((\nabla w^*)t(D\Phi_h)^{-1}\underline{e}_i)((\nabla v_h)^t[(D\Phi_h - I_3)\underline{e}_j]\mathrm{d}x)\right| \\
&+\left|\int_{\Omega_h}\sum_{i,j=1}^{3} a_{ij}^h(x)[(\nabla w^*(D\Phi_h)^{-1}\underline{e}_i)(\nabla v_h\cdot \underline{e}_j]\right. \\
&\left.-(\nabla w^*\cdot \underline{e}_i)(\nabla v_h\cdot \underline{e}_j)]\mathrm{d}x\right|
\end{aligned}
$$

where I_3 is the 3×3 identity matrix. Applying equations (1)-(4) we obtain:

$$
\begin{aligned}
|a(w,\check{v}_h) - a_h(w^*,v_h)| \le{}& Ch^n\|A\|_n\|w\|_{1,\Omega}\|v_h|_{1,\Omega_h} \\
&+\left|\int_{\Omega_h}\sum_{i,j=1}^{3} a_{ij}^h(x)(\nabla w^*(D\Phi_h)^{-1}\cdot \underline{e}_i)(\nabla v_h D(\Phi_h^{-1} - I)\cdot \underline{e}_j)\mathrm{d}x\right| \\
&+\left|\int_{\Omega_h}\sum_{i,j=1}^{3} a_{ij}^h(x)(\nabla w^* D(\Phi_h^{-1} - I)\cdot \underline{e}_i)(\nabla v_h\cdot \underline{e}_j)\mathrm{d}x\right| \\
\le{}& C\|A\|_n\|w\|_{1,\Omega}|v_h|_{1,\Omega_h}(h^n + |D(\Phi_h^{-1} - I)|_{0,\infty,\Omega}) \\
\le{}& Ch^n\|A\|_n\|w\|_{1,\Omega}|v_h|_{1,\Omega_h}\cdot
\end{aligned}
$$

Restricting the inequality equation (6) to the elements of $\mathbf{V}_0^h$ (replacing w with $w_h \circ \Phi_h^{-1}$, $w_h \in \mathbf{V}_0^h$ in equation (6)) we obtain:

$$
|a(\check{w}_h,\check{v}_h) - a_h(w_h,v_h)| \le Ch^n\|A\|_n\|\check{w}_h\|_{1,\Omega}|v_h|_{1,\Omega_h} \tag{9}
$$

$\forall w_h \in \mathbf{V}_0^h$ and $\forall v_h \in \mathscr{V}_h$.

Theorem 1 gives an estimate for the error in discrete solution u_h of the problem $\mathcal{P}_h$.

Theorem 1. Let u and u_h be the solutions of the problems $\mathcal{P}_W$ and $\mathcal{P}_h$ correspondingly. Let also the hypotheses *C1-C3* be valid, the matrix $A = (a_{ij}(x))_{i,j\in\{1,2,3\}}$ be uniformly positive definite in Ω and the triangulation τ_h be n-regular.

Then:

$$|a(u, v_h) - a_h(u_h, v_h) \leq Ch^n\{\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega}\}|v_h|_{1,\Omega_h} \quad \forall v_h \in V_h.$$

Proof. Let P_h: $H_0^1(\Omega) \rightarrow \check{\mathbf{V}}_0^h$ be an orthogonal projection operator with respect to the energy scalar product a$(\cdot, \cdot)$ and $\|\cdot\|_a$ be the corresponding energy norm.

Our first purpose is to estimate the difference $\|u - \check{u}_h\|_{1,\Omega}$. Rewrite the abstract estimate (Brenner and Scott, 1994, p. 210):

$$\|u - \check{u}_h\|_a \leq \|u - P_h u\|_a + \sup_{u_h \in \mathbf{V}_h \setminus \{0\}} \frac{|a(\check{u}_h, \check{v}_h) - a_h(u_h, v_h)|}{\|\check{v}_h\|_a} + \sup_{u_h \in \mathbf{V}_h \setminus \{0\}} \frac{|a(f, \check{v}_h) - (f_h, v_h)_h|}{\|\check{v}_h\|_a}. \tag{10}$$

Applying the inequality (Lemma 8 by Lenoir, 1986)

$$|(f_h, v_h)_h - (f, \check{v}_h)| \leq Ch^n\|f\|_{n,\infty,\Omega}\|\check{v}_h\|_{0,\Omega} \tag{11}$$

and equation (9) to the abstract estimate equation (10) we obtain:

$$\|u - \check{u}_h\|_{1,\Omega} \leq Ch^n\{\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega}\}. \tag{12}$$

Using Cauchy inequality, equation (12) and Lemma 2 we have:

$$\begin{aligned} |a(u, \check{v}_h) - a_h(u_h, v_h)| &\leq |a_h(u_h, v_h) - a_h(u^*, v_h)| + |a_h(u^*, v_h) - a(u, \check{v}_h)| \\ &\leq (\|u_h - u^*\|_{1,\Omega_h} + Ch^n\|A\|_n\|u\|_{1,\Omega})|v_h|_{1,\Omega_h} \\ &\leq (\|u - \check{u}_h\|_{1,\Omega_h} + Ch^n\|A\|_n\|u\|_{1,\Omega})|v_h|_{1,\Omega_h} \\ &\leq Ch^n(\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega})|v_h|_{1,\Omega_h}, \end{aligned}$$

which complete the proof. □

Theorem 2 is the main result in the present paper. The L^2-error in the finite element approximations of the flux across the boundary is estimated by $O(h^{n-1/2})$.

Theorem 2. Suppose that the matrix A is uniformly positive definite in Ω, the hypotheses *C1-C3* hold and the triangulation τ_h is n-regular. Let q and q_h, be the solutions of the problems $\mathcal{F}_W$ and $\mathcal{F}_h$, respectively. Then:

$$\|q - q_h \circ \phi_h^{-1}\|_{0,\Gamma} \leq Ch^{n-1/2}(\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega}). \tag{13}$$

Proof. Let $\mathscr{I}_h$: $H^{1/2}(\Gamma) \rightarrow \mathbf{B}_h$ be a standard interpolation operator on the whole boundary. At first we estimate the difference $q_h - q_I$, where $q_I = \mathscr{I}_h(q \circ \phi_h)$. From the imbedding theorems (Adams, 1975) it follows that $q \in H^{n-1/2}(\Gamma)$ if $u \in H^{n+1}(\Omega)$. Moreover:

$$\|q\|_{n-1/2,\Gamma} \leq C\|u\|_{n+1,\Omega}.$$

Applying the equalities from $\mathcal{F}_W$ and $\mathcal{F}_h$ we obtain:

$$< q_h - q_I, v_h >_h = < q_h - q\circ\phi_h, v_h >_h + < q\circ\phi_h - \mathscr{I}_h(q\circ\phi_h), v_h >_h, \tag{14}$$

$$\begin{aligned} | < q_h - q\circ\phi_h, v_h >_h | &\leq | < q_h, v_h >_h - < q, \check{v}_h > | \\ &\quad + | < q, \check{v}_h > - < q\circ\phi_h, v_h >_h | \\ &\leq |(f_h, v_h)_h - (f, \check{v}_h)| + |a_h(u_h, v_h) - a(u, \check{v}_h)| \\ &\quad + | < q, \check{v}_h > - < q\circ\phi_h, v_h >_h |. \end{aligned} \tag{15}$$

Estimate:

$$\begin{aligned} | < q\circ\phi_h - \mathscr{I}_h(q\circ\phi_h), v_h >_h | &\leq Ch^{n-1/2}\|q\circ\phi_h\|_{n-1/2,\Gamma_h}\|v_h\|_{0,\Gamma_h} \\ &\leq Ch^{n-1/2}\|q\|_{n-1/2,\Gamma}\|v_h\|_{0,\Gamma_h} \end{aligned}$$

by standard interpolation theory and equation (8):

$$| < q, \check{v}_h > - < q\circ\phi_h, v_h >_h | \leq Ch^{n-1/2}\|q\|_{n-1/2,\Gamma}\|v_h\|_{0,\Gamma_h}$$

by Lemma 9 (Lenoir, 1986), the first term in equation (15) by inequality equation (11) and the second term in equation (15) by Theorem 1.

Then:

$$\begin{aligned} | < q_h - q_I, v_h >_h | \leq Ch^{n-1/2}\Big\{ &\|q\|_{n-1/2,\Gamma}\|v_h\|_{0,\Gamma_h} + h^{1/2}(\|A\|_n\|u\|_{n+1,\Omega} \\ &+ \|f\|_{n,\infty,\Omega})\|v_h\|_{1,\Omega_h}\Big\}. \end{aligned}$$

Consider a function $w_h \in \varsigma_h$, which is equal to zero for all internal nodes of the triangulation τ_h. It follows $\|w_h\|_{1,\Omega h} \leq C\|w_h\|_{1/2,\Gamma h}$ and:

$$| < q_h - q_{\mathscr{I}}, w_h >_h | \leq Ch^{n-1/2}(\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega})\|w_h\|0, \Gamma_h. \tag{16}$$

Replacing:

$$w_h = \begin{cases} q_h - q_I & \text{on } \Gamma_h \\ 0 & \text{for all internal nodes of } \tau_h \end{cases}$$

in equation (16) we have:

$$\|q_h - q_I\|_{0,\Gamma_h} \leq Ch^{n-1/2}(\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega}).$$

It remains to use the triangle inequality, equations (8) and (17) to obtain:

$$\begin{aligned} \|q - q_h\circ\phi_h^{-1}\|_{0,\Gamma} \leq C\|q\circ\phi_h - q_h\|_{0,\Gamma_h} &\leq C(\|q\circ\phi_h - q_I\|_{0,\Gamma_h} + \|q_I - q_h\|_{0,\Gamma_h}) \\ &\leq Ch^{n-1/2}(\|A\|_n\|u\|_{n+1,\Omega} + \|f\|_{n,\infty,\Omega}), \end{aligned}$$

which finish the proof. □

5. Conclusion

The proof of $\mathbf{V}_0^h$-ellipticity of $a_h\ (\cdot,\cdot)$ and the comparison between the bilinear forms enable us to prove optimal convergence rate for the consistent isoparametric boundary-flux approximations. General results are obtained for complex domains with Lipschitz-continuous boundary and arbitrary tetrahedral isoparametric elements.

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